∘ Block encoding of sparse matrices

<u>Preliminaries:</u> $\mathcal{H}$, hilbert space $\quad \mathcal{H} \cong \mathbb{C}^N,\ N = 2^n,\ n := \#$ qubits

$$L(\mathcal{H}) = \mathbb{C}^{N \times N}$$

$$v \in \mathbb{C}^N, \quad [N] = \{0, \dots, N-1\}, \quad v = \begin{pmatrix} v_0 \\ \vdots \\ v_{N-1} \end{pmatrix}$$

$$\|v\|_2 = \|v\| := \sqrt{\sum |v_i|^2}$$

$|v\rangle = \frac{v}{\|v\|}$ (normally) (not consistent notation during)

unnormalized vector: $|v\rangle$

$A \in \mathbb{C}^{M \times N} \qquad A = [a_{ij}]_{i \in [M], j \in [N]}$

$A^T \qquad \overline{A} \equiv A^* = [\overline{a_{ij}}]_{ij} \qquad A^\dagger = \overline{(A^T)}$

$\langle v| = (\overline{v_0}, \dots, \overline{v_{N-1}})$ (hermitian of ket)

$A \in \mathbb{C}^{N \times N}$
- Hermitian: $A = A^\dagger$
- Normal: $AA^\dagger = A^\dagger A$
- Unitary: $A^\dagger A = I = AA^\dagger$

~~THINK: A is Hermitian & Unitary~~

Positive Semidefinite (PSD) $\quad A \succeq 0$
Positive definite (PD) $\quad A \succ 0$

2 norm has induced norm $\|A\|_2$

operator norm $\quad \|A\|_2 = \sup_{\|v\|_2 = 1} \|Av\|_2$

Schatten-p norm $\quad \|A\|_p := \left(Tr\left[(A^\dagger A)^{\frac{p}{2}}\right]\right)^{1/p}$ (kind of weird)

Cases (important): $p = 1 \quad \|A\|_1 = Tr(\sqrt{A^\dagger A})$ (trace norm)
(different induced 1-norm in linalgebra)

Schatten-p norm

$p=\infty$ $\quad \|A\|_\infty$ (need to take lim)

$$\underbrace{\|A\|_\infty = \|A\|}_{\text{THINK (SVD)}} \quad \text{(same as operator norm)}$$

$p=2$ $\quad \|A\|_2 = \left(\mathrm{Tr}(A^\dagger A)\right)^{1/2}$ $\quad$ (Frobenius norm)

$\mathcal{H} = \mathbb{C}^N$ $\quad$ braket = inner product $\quad \psi, \varphi \in \mathcal{H} \quad \langle \psi | \varphi \rangle = \sum \bar{\psi}_i \varphi_i$

ketbra = outer product $\quad \cdots \quad |\psi\rangle\langle\varphi| = |\psi\rangle\langle\varphi| \in \mathcal{L}(\mathcal{H})$

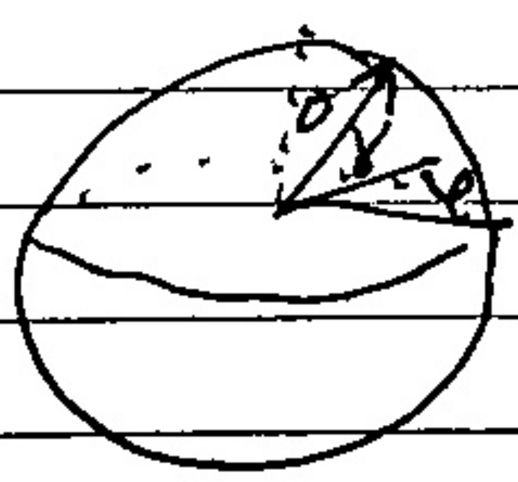
$$(|\psi\rangle\langle\varphi|)(|u\rangle) = |\psi\rangle \cdot (\langle\varphi|u\rangle) \in \mathbb{C}$$

Ex. single qubit Bloch sphere $\mathcal{H} = \mathbb{C}^2$

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$|\psi\rangle = a|0\rangle + b|1\rangle \qquad |a|^2+|b|^2=1$$

$$= \underbrace{e^{i\delta}}_{\text{global phase}}\left(\cos\tfrac{\theta}{2}|0\rangle + e^{i\varphi}\sin\tfrac{\theta}{2}|1\rangle\right) \qquad 0\le\theta\le\pi,\ 0\le\varphi\le 2\pi$$



Pauli Matrices,

$$X = \begin{bmatrix}0&1\\1&0\end{bmatrix} \qquad Y = \begin{bmatrix}0&-i\\i&0\end{bmatrix} \qquad Z = \begin{bmatrix}1&0\\0&-1\end{bmatrix} \qquad \sigma_{X,Y,Z}$$

Hadamard gate $\quad H = \frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}$ $\quad$ (there is also "Hamiltonian" gates, so be careful about notation)

Hadamard (H), Hamiltonian $(\mathcal{H})$

Pauli: ...
Hadamard ..

Thm (Spectral thm of normal matrices) $A \in \mathbb{C}^{N \times N}$ normal
$\iff A = VDV^{\dagger}$ $V \in U(N)$, D diagonal

when A is Hermitian, D entries are all real

Thm (SVD) $A \in \mathbb{C}^{M \times N}$
$A = U \Sigma V^{\dagger}$ $U \in U(M)$, $V \in U(N)$, Σ diag + zeroes

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Observable (means Hermitian matrix)
$\mathcal{O} \in \mathbb{C}^{N \times N}$ is Hermitian

$\mathcal{O} = \sum_{i \in [N]} \lambda_i P_i$ $\lambda_i \in \mathbb{R}$, $P_i^2 = P_i$ projection
$P_i = |v_i\rangle\langle v_i|$

(observed means there is supernatural/natural process where we get: λ_i)

① λ_i w.p. $p_i = \| P_i |\psi\rangle \|^2$
② collapse $|\psi\rangle \longrightarrow \frac{P_i |\psi\rangle}{\| P_i |\psi\rangle \|}$

Ex $\mathcal{O} = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$X = |+\rangle\langle +| - |-\rangle\langle -|$
$\lambda_0 = +1$ $\lambda_1 = -1$
$v_0 = |+\rangle = 1/\sqrt{2} \, (|0\rangle + |1\rangle)$
$v_1 = |-\rangle = 1/\sqrt{2} \, (|0\rangle - |1\rangle)$

$|\psi_0\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} (|+\rangle + |-\rangle)$
$|+\rangle$ w.p. $1/2$
$|-\rangle$ w.p. $1/2$

Tensor Product ...

$\mathcal{H}_1 \otimes \mathcal{H}_2$

product state $|u\rangle \in \mathcal{H}_1,\ |v\rangle \in \mathcal{H}_2 \quad |u\rangle \otimes |v\rangle \equiv |u,v\rangle \equiv |u\rangle|v\rangle$

$|u_1\rangle, |u_2\rangle \in \mathcal{H}_1$
$|v_1\rangle, |v_2\rangle \in \mathcal{H}_2$

$\langle u_1, v_1 | u_2, v_2\rangle := \langle u_1|u_2\rangle \cdot \langle v_1|v_2\rangle$ (extended by linearity)

Ex. 2-qubit system $(\mathbb{C}^2) \otimes (\mathbb{C}^2) \cong \mathbb{C}^4$

(convention row major order, last index first)

$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ etc...

CNOT (2-qubit unitary): $|a\rangle|b\rangle \equiv$

$CNOT\,|a\rangle|b\rangle = |a\rangle|a \oplus b\rangle$

$CNOT\,|0\rangle|b\rangle = |0\rangle|b\rangle$
$CNOT\,|1\rangle|b\rangle = |1\rangle|1 \oplus b\rangle$

Controlled-U: $CU = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U \longleftrightarrow CU = \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix}$

$CNOT = CX$ THINK: expand ops

Quantum Circuit (diagram)

$|\psi\rangle$ —[U]— $U|\psi\rangle$

Ex $|0\rangle$ —[X]— $|1\rangle$ $\quad$ $|0\rangle$ —[H]— $|+\rangle$

Ex CNOT

($\oplus$ means X)

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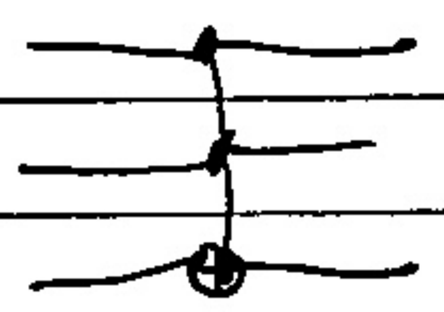
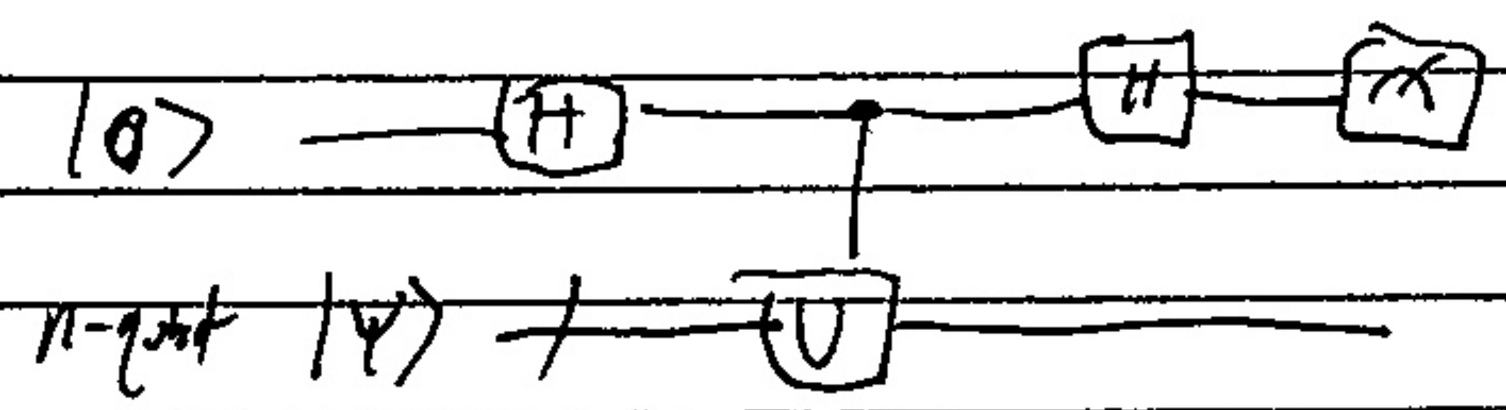
Quantum Circuit...

Toffoli  $\in \mathbb{C}^{8\times 8}$ (THINK! Write down matrix rep)

Register (uses classical bits) .. Quantum Registers

$|0^n\rangle = |0\rangle^{\otimes n}$

Ex. Hadamard test circuits



(~~measurement with respect to~~ Z, i.e. in the same computational basis)

(look at units for computation)

Q: what is the probability of measuring 0?

Quantum Computing (start classical, do something with unitary, convert to computational basis, then measurement, post processing detecting useful info)
(No other model away from this)