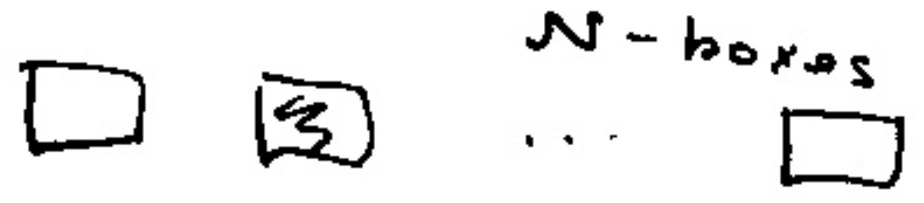
2/10

Grover's algorithm
qubitization w. basis change
amplitude amplification (AA)
oblivious amplitude amplification



$f: \{0,1\}^n \to \{0,1\}$ (where 1 is marked)

$f(x) \quad R_{x_0} = I - 2|x_0\rangle\langle x_0|$

phase kick back

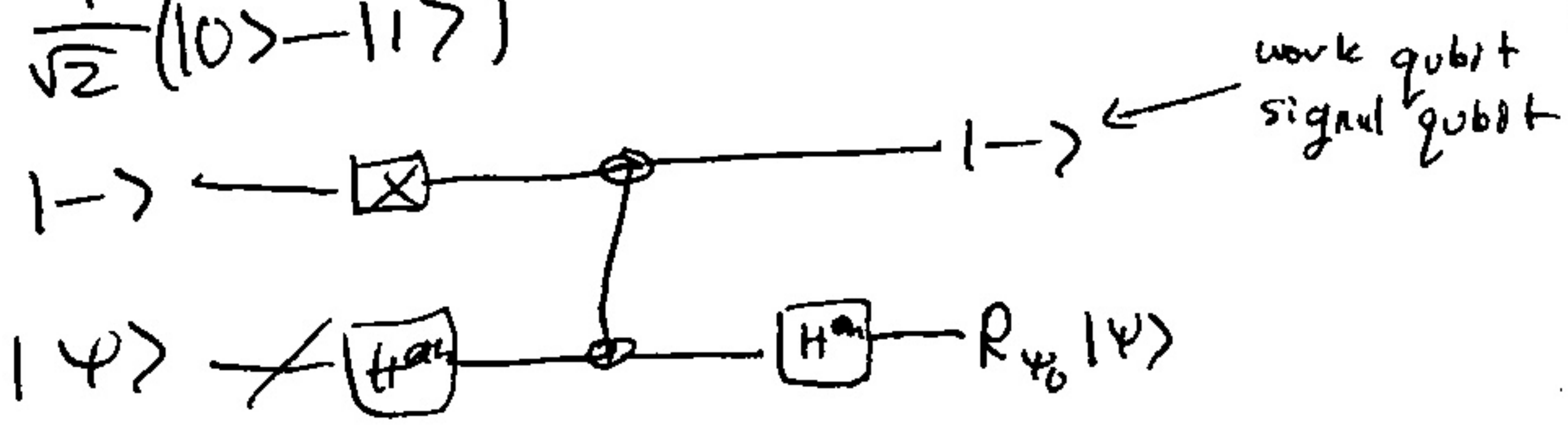
$$R_{x_0}|x\rangle = (-1)^{f(x)}|x\rangle = \begin{cases} -|x_0\rangle & x = x_0 \\ |x\rangle, & \text{other} \end{cases}$$

How many times do we need to query R_{x_0} (quantum unboxing)

$$|\psi_0\rangle = \frac{1}{\sqrt{2^n}} \sum_{n=0}^{2^n-1} |i\rangle = H^{\otimes n}|0^n\rangle$$

$$R_{\psi_0} = 2|\psi_0\rangle\langle\psi_0| - I = H^{\otimes n}(2|0^n\rangle\langle 0^n| - I)H^{\otimes n}$$

$\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$



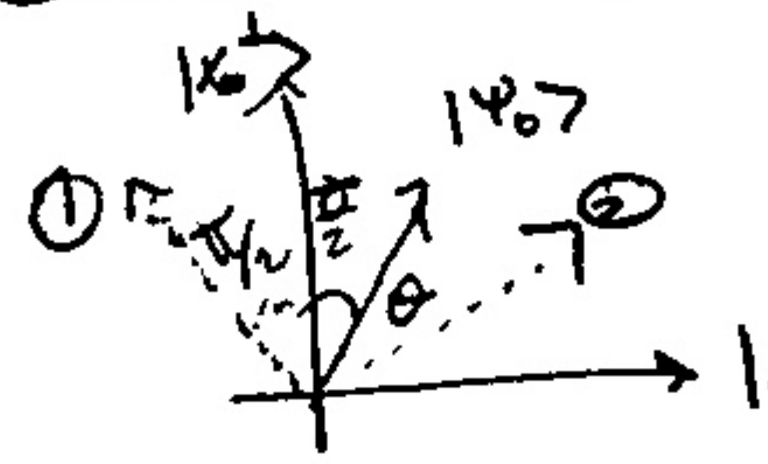
Grover Iterate: $G = R_{\psi_0} R_{x_0}$

claim $G^k|\psi_0\rangle \quad k \sim \Theta(\sqrt{N})$

Measure in computational basis

then w. $\Omega(1)$ prob find $|x_0\rangle$

Geometric Perspective:



angle $\frac{\theta}{2} \to \frac{3\theta}{2}$

$2\theta + \frac{\theta}{2} = \frac{5\theta}{2}$

$|x_0^\perp\rangle = \frac{1}{\sqrt{N-1}} \sum_{x \neq x_0} |x\rangle$

$|\psi_0\rangle = \frac{1}{\sqrt{N}}|x_0\rangle + \sqrt{\frac{N-1}{N}} \cdot |x_0^\perp\rangle$

$= \sin\frac{\theta}{2}|x_0\rangle + \cos\frac{\theta}{2}|x_0^\perp\rangle$

$\mathcal{B} = \{|x_0\rangle, |x_0^\perp\rangle\}$

$$G^k|\psi_0\rangle = \sin\left(\frac{2k+1}{2}\theta\right)|x_0\rangle + \cdots$$

Which k? $\frac{2k+1}{2}\theta \approx \frac{\pi}{2} \Longrightarrow k \approx \left(\frac{\pi}{\theta} - 1\right) \cdot \frac{1}{2}$

$\left(\theta \approx \frac{2}{\sqrt{N}}\right) \approx \frac{\pi\sqrt{N}}{4}$

Grover's algorithm: "doesn't converge" $\lessgtr$

M mark vertices $\quad f(x) = \begin{cases} 1, & x \in \text{marked set} \\ 0, & \text{else} \end{cases}$

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qubitization $\quad U_A^\dagger Z_\pi U_A Z_\pi$

$A \in \mathbb{C}^{M \times N}$

$$\mathcal{B} = \{ \underbrace{\overbrace{|\varphi_0\rangle, \dots, |\varphi_{N-1}\rangle}^{\text{projector } \Pi}, \dots}_{M > N} \} \qquad \mathcal{B}' = \{ \overbrace{|\psi_0\rangle, \dots, |\psi_{N-1}\rangle}^{\text{projector } \Pi'}, \dots \}$$

$$\Pi' U_A \Pi = \sum_{i,j \in [N]} |\psi_i\rangle A_{ij} \langle \varphi_j| \qquad U_A = \begin{bmatrix} A & * \\ * & * \end{bmatrix}$$

"more general version of block encoding"

~~coefficient~~ $\cancel{\tilde{U}_A = \dots}$

coefficient $\tilde{U}_A = [U_A]_{\mathcal{B}}^{\mathcal{B}'}$

$$\tilde{U}_A^\dagger \; Z_{\Pi_0 m} \; \tilde{U}_A \; Z_{\Pi_0 n}$$

— [lec 10.6]

Interpret Grover's alg using qubitization

$\mathcal{B} = \{ |\varphi_0\rangle, \dots \rangle \qquad \mathcal{B}' = \{ |x_0\rangle, \dots \}$

"odd order polynomial"

$U_A = R_{\varphi_0} \qquad [U_A]_{\mathcal{B}}^{\mathcal{B}'} = \begin{bmatrix} a & * \\ * & * \end{bmatrix} \qquad a = \sin\frac{\theta}{2} = \frac{1}{\sqrt{N}}$

$Z_\Pi = R_{\varphi_0} \qquad Z_{\Pi'} = -R_{x_0}$

$U_A Z_\Pi (U_A^\dagger Z_{\Pi'} U_A Z_\Pi)^k \qquad U_A Z_\Pi = R_{\varphi_0} R_{\varphi_0} = I$

$$\parallel$$

$$(U_A^\dagger Z_{\Pi'})^k = (-1)^k (R_{\varphi_0} R_{x_0})^k$$

$$\to \begin{bmatrix} (-1)^k T_{2k+1}(a) & * \\ * & * \end{bmatrix} \qquad \cancel{(-1)^k} \sin\left(\frac{(2k+1)}{2}\theta\right)|x_0\rangle + |\perp\rangle$$

$$T_{2k+1}(a) = (-1)^k \sin((2k+1) \arcsin a)$$

$\star \; |\psi\rangle_0 = \cancel{\dots} \, T_{2k+1}(a) \, |x_0\rangle\langle\varphi_0|$

"AA and AE paper"

$U_{\Psi_0}|0^n\rangle = |\Psi_0\rangle$

$|\Psi_0\rangle = \sqrt{p}\,|\Psi_{good}\rangle + \sqrt{1-p}\,|\Psi_{bad}\rangle$

solution $|\Psi_{good}\rangle$ w. $(o) + O(\frac{1}{\sqrt{p}})$

$R_{good} = I - 2|\Psi_{good}\rangle\langle\Psi_{good}|$

$G = R_{\Psi_0} R_{good}$

G^k $\quad k \sim \frac{\pi}{4\sqrt{p}}$

$|\Psi_{good}\rangle = |0^m\rangle|\Psi_{good}\rangle$

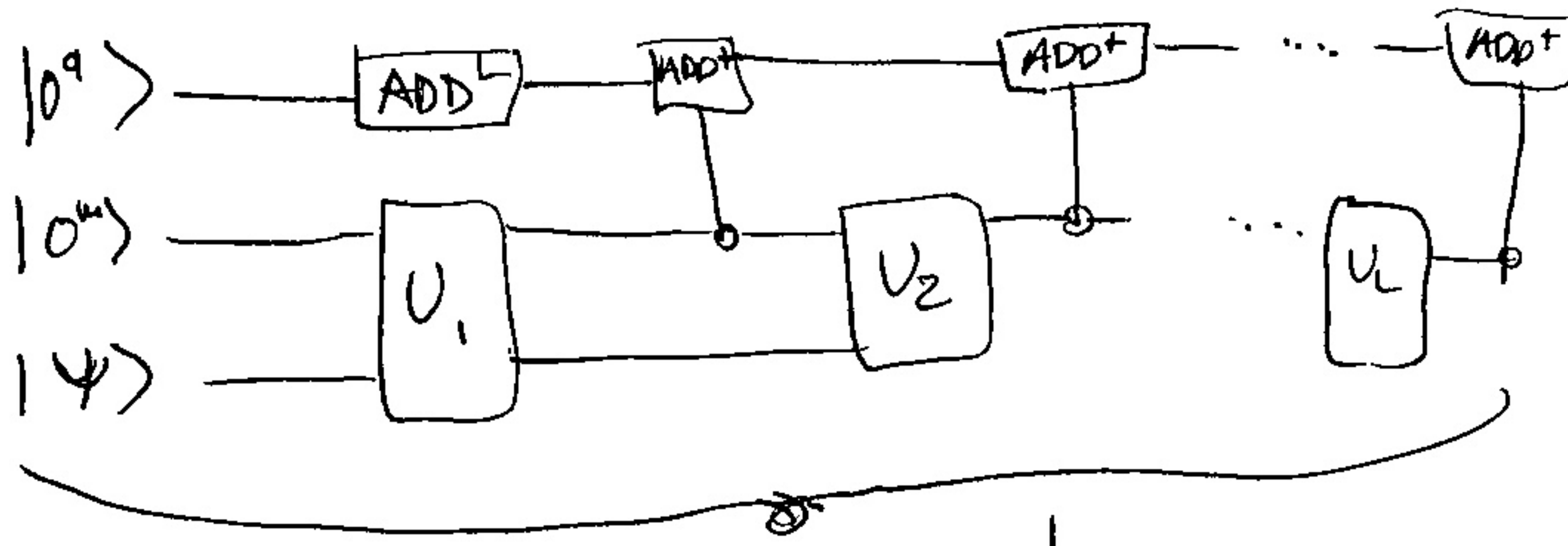
Revisit composition of Block Encodings

$U_1, \dots, U_L$ $\quad U_i \in BE_{\alpha_i, m}(A_i)$ $\quad$ goal: block encoding of $A_L, \dots, A_1$

Naive way: use $m \cdot L$ qubits $\quad$ subnormalization $\prod \alpha_i$

more efficient: L additional qubits $\quad$ subnormalization $\prod_i \alpha_i$

even more efficient: $\lceil \log L \rceil$ " " " "



$p = \left(\frac{\|A_L \cdots A_1|\Psi\rangle\|}{\prod_i \alpha_i}\right)^2$

$\frac{A_L \cdots A_1|\Psi\rangle}{\| \quad \|}$

U_{Ψ_0} $\quad$ w. $\frac{1}{\sqrt{p}}$

Standard  w.p. p

repeat $1/p$ times

 ← layer depth

$1/\sqrt{p}$

repeat $O(1)$

"AA and AE paper"

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$|\psi_0\rangle = \sqrt{p}\,|\psi_{good}\rangle + \sqrt{1-p}\,|\psi_{bad}\rangle$

solution $|\psi_{good}\rangle$ w. $(o) + \mathcal{O}\left(\frac{1}{\sqrt{p}}\right)$

$R_{good} = I - 2|\psi_{good}\rangle\langle\psi_{good}|$

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$G^k \qquad k \sim \frac{\pi}{4\sqrt{p}}$

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2/10

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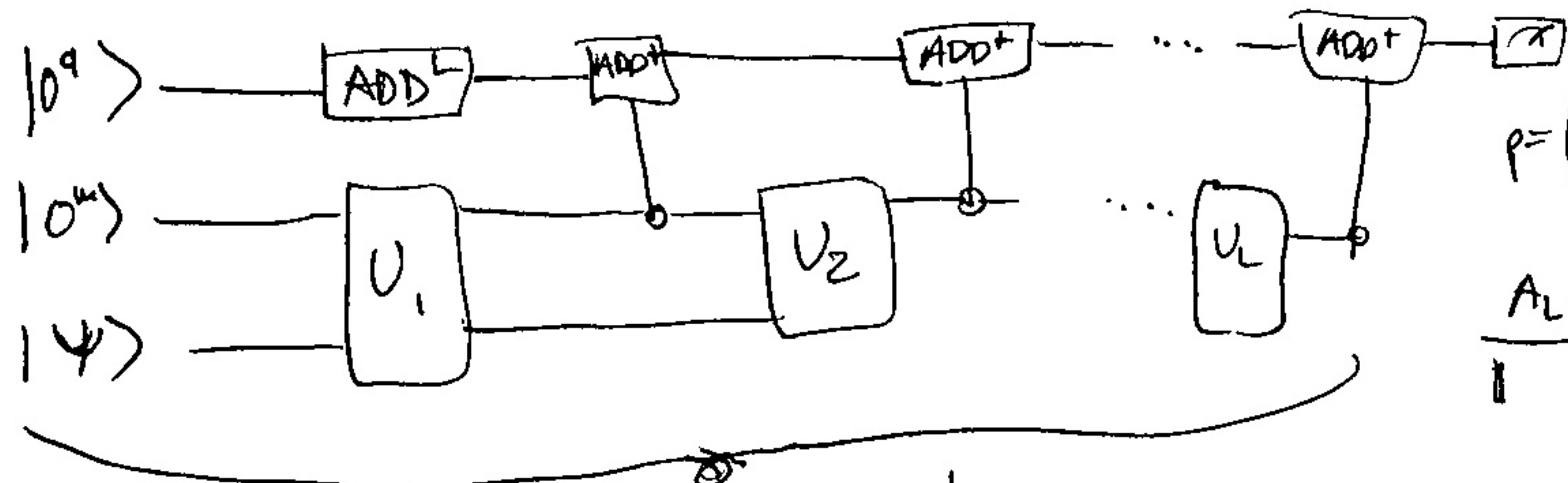

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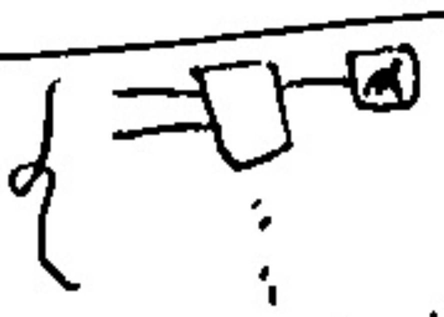
$\frac{A_L \cdots A_1|\psi\rangle}{\| \quad \|}$

$U_{\psi_0} \quad$ w. $\frac{1}{\sqrt{p}}$

standard



w.p. p

repeat $1/p$ times

layer depth

$1/\sqrt{p}$

repeat $\mathcal{O}(1)$

oblivious AA

special case: block encoding of a unitary matrix $\qquad e^{iHt}$

$$V = \begin{pmatrix} \frac{U}{\alpha} & * \\ * & * \end{pmatrix} \qquad V \in BE_{\alpha,m}(U)$$

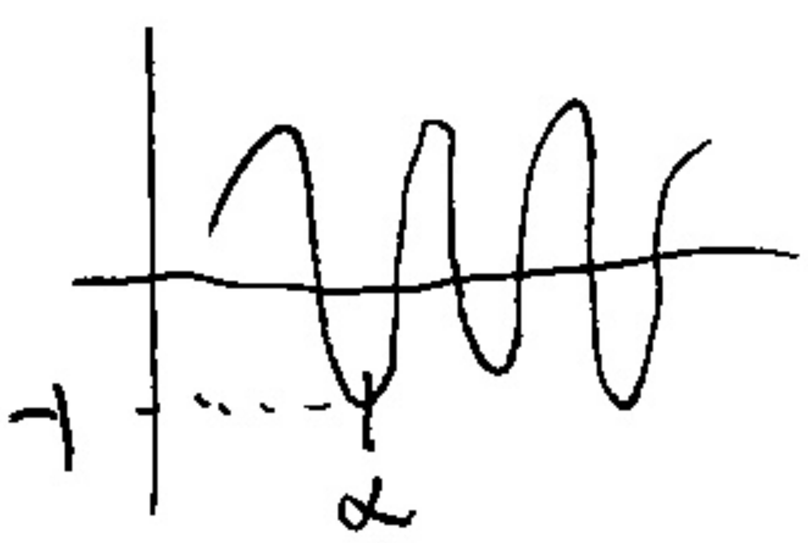
$$V|0^m\rangle|\psi\rangle = \frac{1}{\alpha}|0^m\rangle U|\psi\rangle + |\perp\rangle$$

$$A = \frac{U}{\alpha} = U \cdot \underbrace{\frac{1}{\alpha} \cdot I}_{\Sigma}$$

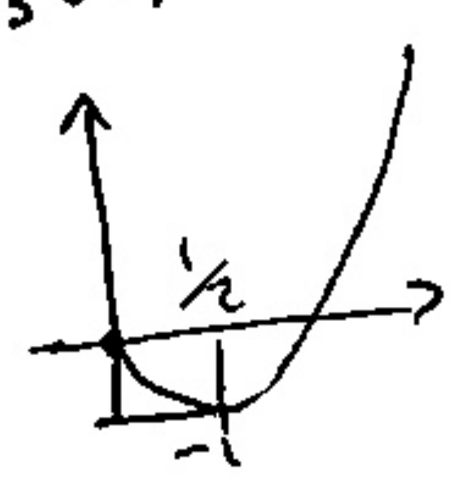
$$T_{2k+1}^{\diamond}(A) \qquad \cancel{T_{2k}^{\diamond}(A)}$$

$$= U\,T_{2k+1}\!\left(\tfrac{1}{\alpha}\right) I$$

$$= U$$



$$T_3(x) = 4x^3 - 3x$$



"quantum channel"

$\|\cdot\|$