Quantum Signal Processing + nonlinear fourier transform 2/19

$O(x) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $\quad [O(x)]^d = \begin{bmatrix} T_k(x) & * \\ * & * \end{bmatrix}$

$x \in [-1,1]$, $\theta = \arccos x$

$$e^{i\phi_0 Z} O(x) e^{i\phi_1 Z} O(x) \cdots e^{i\phi_{d-1} Z} O(x) e^{i\phi_d Z} = \begin{bmatrix} P(x) & -Q(x)\sqrt{1-x^2} \\ Q(x)\sqrt{1-x^2} & \overline{P}(x) \end{bmatrix}$$

$\Phi = (\phi_1, \ldots, \phi_d)$ phase factors

(1) $\deg P \le d$ $\quad \deg Q \le d-1$ $\quad P, Q \in \mathbb{C}[x]$

(2) parity P $d \bmod 2$, q - $(d-1) \bmod 2$

(3) (normalization) $|P(x)|^2 + (1-x^2)|Q(x)|^2 = 1$ $\quad x \in [-1,1]$

"layer stripping"
Shor 1917

---

target function $F = \mathrm{Re}\, P \in \mathbb{R}[x]$ (or $F = \mathrm{Im}\, P$)

(1) $\deg F \le d$

(2) parity F $d \bmod 2$

(3) $\|F\|_\infty := \sup_{x \in [-1,1]} |F(x)| \le 1$ "fully coherent"

$F^{SV}(A)$

$\left(\frac{F}{\alpha}\right)^{SV}(A)$

write $P(x) = F(x) + iG(x)$ $\quad Q(x) \in \mathbb{C}[x]$, $F, G \in \mathbb{R}[x]$

~~$|F(x)|^2+$~~ $|G(x)|^2 + (1-x^2)|Q(x)|^2 = 1 - |F(x)|^2$ ← find all roots

poly of deg 2d

complementary polynomials → layer stripping

~~Tuesday~~ • showing slide on board

sym choice ⟷ $Q \in \mathbb{R}[x]$

"qsppack github"

2/19

Simplest alg for finding phase factors: <u>Fixed point iteration</u>

<u>d even</u>: $F(x) = \sum^{d/2} c_k T_{2k}(x)$ $\vec{c} = (c_0, \dots, c_{\frac{d}{2}})$

sym phase factors $\Phi_{sym} = (\frac{\phi_0}{2}, \phi_1, \dots, \phi_{\frac{d}{2}})$

$(\phi_{\tilde{d}}, \dots, \phi_1, \phi_0, \phi_1, \phi_2, \dots, \phi_{\tilde{d}})$

Define $\mathcal{F}(\Phi) = \vec{c}$

Goal: $\vec{c}$. $\mathcal{F}^{-1}(\vec{c})$

Fact: $\mathcal{F}(0) = \vec{0}$

$\nabla \mathcal{F}(\vec{0}) = 2\vec{I}$

Alg:

$$\begin{cases} \Phi^{(l+1)} = \Phi^{(l)} - \frac{1}{2}\left(\mathcal{F}(\Phi^{(l)}) - \vec{c}\right) \\ \Phi^{(0)} = \vec{0} \end{cases}$$

($\frac{1}{2} \to [\nabla \mathcal{F}(\vec{0})]^{-1}$)

"Newton's method"

<u>Thm</u> $\|c\|_1 \lesssim 0.861$ then FPI converges ??

(Linear) Fourier Transform. $\mathbb{T}$: unit circle $z \in \mathbb{T}$ $z = e^{i\theta}$

$f(x) = \sum_k c_k T_k(x)$ $x = \frac{z+z^{-1}}{2}$, $z = e^{i\theta}$

$= \cos\theta$

↓

$F(z) = \sum_{\ell=m}^{n} d_\ell z^\ell$

$F_0(z) = 0$, $F_{\ell+1}(z) = F_\ell(z) + d_\ell z^\ell$

Nonlinear Fourier Transform on <u>SU(2)</u> ~~$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$~~ $\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$ unitary, det=1

$F_0(z) = I$ $F_\ell(z) = F_\ell(z) \begin{pmatrix} 1 & \gamma_\ell z^\ell \\ -\bar{\gamma}_\ell z^{-\ell} & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{1+|\gamma_\ell|^2}}$

$\vec{\gamma} = (\gamma_m, \dots, \gamma_n)$ compactly supported setting

$$\vec{\gamma} = \prod_{k=m}^{n} \frac{1}{\sqrt{1+|\gamma_k|^2}} \begin{pmatrix} 1 & \gamma_k z^k \\ -\bar{\gamma}_k z^{-k} & 1 \end{pmatrix}$$

when $\|\vec{\gamma}\|_2$ is small

$$\approx \begin{pmatrix} 1 & \sum_{k=m}^{n} \gamma_k z^k \\ -\sum_{k=m}^{n} \bar{\gamma}_k z^k & 1 \end{pmatrix}$$

~~Schur 19~~

Schur 1917

f analytic in $\mathbb{D} = \{z \; (|z|<1)\}$

what are the conditions s.t.

$f: \mathbb{D} \to \mathbb{D}$

Schur's alg

→ Schur complement

NLFT on $SU(1,1)$

$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$(\vec{u}, \vec{v}) = \bar{u}_1 v_1 - \bar{u}_2 v_2$

"show slides"

$$\int_{\mathbb{T}} |g|^2 := \frac{1}{2\pi} \int_0^{2\pi} |f(\cos\theta)|^2 d\theta = \frac{2}{\pi} \int_0^1 \frac{|f(x)|}{\sqrt{1-x^2}} dx$$

$g(e^{i\theta}) = f(\cos\theta)$

$= := \|f\|_S^2$ Szegő norm

$$\lim_{d\to\infty} \|p_d - f\|_S = 0$$