Qubitization II

(CS) cosine–sine decomposition

s-sparse bounded $C(j,l) = j + l - \dots$

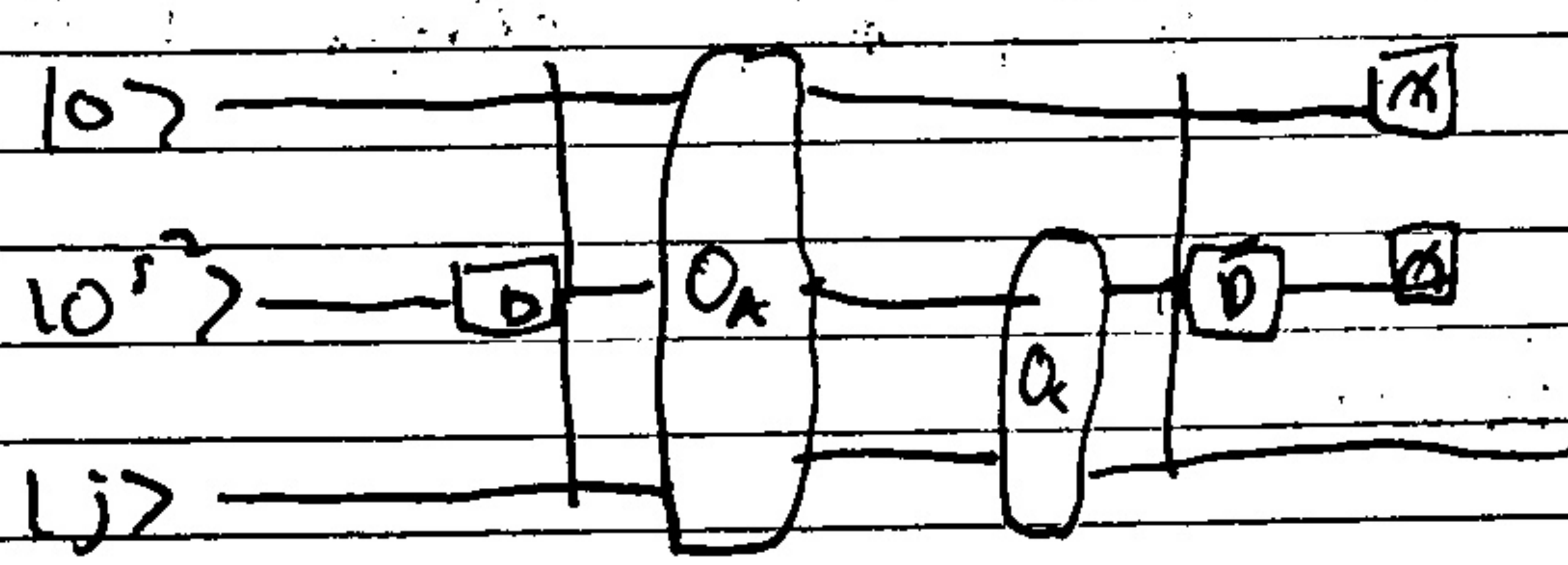
$O_C |l\rangle|j\rangle = |l\rangle|C(j,l)\rangle$

$O_A |0\rangle|l\rangle|j\rangle = (A_{C(j,l),j}|0\rangle + \sqrt{1-|A_{C(j,l),j}|^2}|1\rangle)|l\rangle|j\rangle$

$A = \sum_{l\in[s]} A^{(l)}$

$\|\alpha\|_1 = s$

$s = 2^{s^2}$



$\frac{1}{\sqrt{s}}\sum_{l\in[s]}|l\rangle$

$|l\rangle|j\rangle \to |l\rangle|0\rangle$

$|0\rangle|0\rangle \to |0\rangle|0\rangle$

$|1\rangle|0\rangle \to |1\rangle|1\rangle$

$|0\rangle|1\rangle \to |0\rangle|1\rangle$

$|1\rangle|1\rangle \to |1\rangle|0\rangle$

$U_A \in HBE_{1,m}(A)$

$U_A = \begin{bmatrix} A & * \\ * & * \end{bmatrix}$ $\cancel{U_A^\dagger U_A} = \begin{bmatrix} A & * \\ * & * \end{bmatrix}$

A Hermitian $U_A \in BE_{1,m}(A)$ $U_A^\dagger = \begin{bmatrix} A & * \\ * & * \end{bmatrix} \neq U_A$

called a prime

$U_A|0^m\rangle|v_i\rangle = \lambda|0^m\rangle|v_i\rangle + \sqrt{1-\lambda_i^2}\,|\perp_i'\rangle$

because applies:

$U_A^\dagger|0^m\rangle|v_i\rangle = \lambda_i|0^m\rangle|v_i\rangle + \sqrt{1-\lambda_i}\,|\perp_i\rangle$

$|0^m\rangle|v_i\rangle = \lambda_i(\lambda_i|0^m\rangle|v_i\rangle + \sqrt{1-\lambda_i^2}\,|\perp_i\rangle) + \sqrt{1-\lambda_i^2}\,U_A^\dagger|\perp_i'\rangle$

$U_A^\dagger|\perp_i'\rangle = \sqrt{1-\lambda_i^2}\,|0^m\rangle|v_i\rangle - \lambda_i|\perp_i\rangle$

$U_A|\perp_i\rangle = \sqrt{1-\lambda_i^2}\,|0^m\rangle|v_i\rangle - \lambda_i|\perp_i'\rangle$

$\mathcal{B}_i = \{|0^m\rangle|v_i\rangle, |\perp_i\rangle\}$ $\mathcal{B}_i' = \{|0^m\rangle|v_i\rangle, |\perp_i'\rangle\}$

$\mathrm{Span}\,\mathcal{B}_i \underset{U_A^\dagger}{\overset{U_A}{\rightleftarrows}} \mathrm{Span}\,\mathcal{B}_i'$

"dessert course

$$\left[U_A\right]_{\beta_i}^{\beta_i'} = \begin{bmatrix} \lambda_i & \sqrt{1-\lambda_i^2} \\ \sqrt{1-\lambda_i^2} & -\lambda_i \end{bmatrix} = \left[U_A^\dagger\right]_{\beta_i'}^{\beta_i}$$

$$Z_\Pi = (2|0^m\rangle\langle 0^m| - I) \otimes I$$

<u>Qubitization iterate</u> $O_A = U_A^\dagger Z_\Pi U_A Z_\Pi, \in BE_{1,m}(T_2(A))$

$$O_A^k \in BE_{1,m}(T_{2k}(A))$$

$$U_A Z_\Pi O_A^k \in BE_{1,m}(T_{2k+1}(A))$$

"dealing with eigenvalue transformations is incomplete, ongoing research"

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singular value transformations

$$U_A = \begin{bmatrix} W\Sigma V^\dagger & * \\ * & * \end{bmatrix}$$

$$A = W\Sigma V^\dagger = \sum_i |w_i\rangle\langle v_i| \cdot \sigma_i$$

$$\sigma_i \in [0,1]$$

$$U_A^\dagger = \begin{bmatrix} V\Sigma W^\dagger & * \\ * & * \end{bmatrix}$$

$$U_A |0^m\rangle|v_i\rangle = \lambda_i |0^m\rangle|w_i\rangle + \sqrt{1-\lambda_i^2}\,|\perp_i'\rangle$$

$$U_A^\dagger |0^m\rangle|w_i\rangle = \lambda_i |0^m\rangle|v_i\rangle + \sqrt{1-\lambda_i^2}\,|\perp_i\rangle$$

$$\operatorname{span} \beta_i \underset{U_A^\dagger}{\overset{U_A}{\rightleftarrows}} \operatorname{span} \beta_i'$$

"we are doing SV-transformations rather than eigenvalue transformations"

$A = W \Sigma V^\dagger \qquad f^{sv}(A) = W f(\Sigma) V^\dagger$

$f^{\triangleright}(A) = V f(\Sigma) V^\dagger$

Thm (CS-decomposition) Let $q \geq p$, $U \in \mathbb{C}^{(p+q)\times(p+q)}$ unitary

$$U = \begin{array}{c} \\ p \\ q \end{array}\begin{array}{c} \begin{array}{cc} p & q \end{array} \\ \left[\begin{array}{c|c} W_1 & \\ \hline & W_2 \end{array}\right] \end{array} \begin{array}{c} \\ p \\ p \\ q-p \end{array}\begin{array}{c} \begin{array}{ccc} p & p & q-p \end{array} \\ \left[\begin{array}{c|c|c} C & S & \\ S & -C & \\ \hline & & I_{q-p} \end{array}\right] \end{array} \begin{array}{c} \begin{array}{cc} p & q \end{array} \\ \left[\begin{array}{c|c} V_1^\dagger & \\ \hline & V_2^\dagger \end{array}\right] \end{array}$$

$$C = \begin{bmatrix} \ddots & & \\ & \cos\theta_i & \\ & & \ddots \end{bmatrix} \qquad S = \begin{bmatrix} \ddots & & \\ & \sin\theta_i & \\ & & \ddots \end{bmatrix}$$

$0 \leq \theta_i \leq \frac{\pi}{2}$

CS → qubitization

$$U_A = \underbrace{\begin{array}{c} \\ N \\ \\ \end{array}\begin{array}{c} N \\ \left[\begin{array}{c|c} A & \\ \hline & \\ & \end{array}\right] \end{array}}_{M \cdot N} = \left[\begin{array}{c|c} W_1 & \\ \hline & W_2 \end{array}\right]$$

$$\to \left[\begin{array}{c|c|c} \Sigma & \sqrt{1-\Sigma^2} & 0 \\ \sqrt{1-\Sigma^2} & -\Sigma & 0 \\ \hline 0 & 0 & I \end{array}\right] \left[\begin{array}{c|c} V_1^\dagger & \\ \hline & V_2^\dagger \end{array}\right]$$

Grover alg. Unstructured search

□ . . . ■ . . . □ N boxes

x_0

U_{x_0} $f : [N] \to \{0,1\}$

$\downarrow$ bit oracle $N = 2^n$

$$V_f |x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle, \quad x \in \{0,1\}^n, \ y \in \{0,1\}$$

"Shadbanmike, Aaronson"

$$R_{x_0}|x\rangle = \begin{cases} |x\rangle & x \neq x_0 \\ -|x_0\rangle & x = x_0 \end{cases}$$

$$R_{x_0}|x\rangle = \underbrace{(-1)^{f(x)}}_{\text{phase}} |x\rangle$$

$R_{x_0} = I - 2|x_0\rangle\langle x_0|$ $\longleftarrow$ debute

$$|\psi_0\rangle = \frac{1}{\sqrt{N}} \sum |i\rangle = H^{\otimes n}|0^n\rangle$$

$$R_{\psi_0} = 2|\psi_0\rangle\langle\psi_0| - I$$

Grover iterate $G = R_{\psi_0} R_{x_0}$

Claim: $G^k|\psi_0\rangle \xrightarrow{\text{measure}} |x_0\rangle$

$k \sim \sqrt{N}$

$= N^{\frac{1}{2} - \alpha}$